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ASYMPTOTIC EXPANSIONS OF TRACES FOR CERTAIN CONVOLUTION OPERATORS

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ABSTRACT. A version of Szegö's theorem in Euclidean space gives the first two terms of the asymptotics as $\alpha \to \infty$ of the determinant of convolution operators on $L_2(\alpha\Omega)$, where Ω is a bounded subset of $\mathbf{R}^n$ with smooth boundary. In this paper the more general problem of the asymptotics of traces of certain analytic functions of the operators is considered and the next term in the expansion is obtained.

1. Introduction. In 1960 H. Widom [**6**] discovered the following analogue of theorems of G. Szegö [**2**, §5.5] and M. Kac [**3**]. Consider the operator

$$T_\alpha : f \to \int_{\alpha\Omega} k(x-y) f(y)\, dy \tag{1.1}$$

on $L_2(\alpha\Omega)$, where Ω is a compact subset of $\mathbf{R}^n$ with C^1 boundary and α is a real parameter. With

$$\sigma(\xi) = \hat{k}(\xi) = \int_{\mathbf{R}^n} k(z) e^{-i\xi\cdot z}\, dz,$$

$$g^\vee(z) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} g(\xi) e^{i\xi\cdot z}\, d\xi \quad \text{and} \quad s(z) = (\log(1+\sigma))^\vee(z)$$

Widom proved that under suitable conditions

$$\log\det(1+T_\alpha) = a_0\alpha^n + a_1\alpha^{n-1} + o(\alpha^{n-1}), \qquad \alpha \to \infty,$$

where

$$a_0 = \operatorname{vol}(\Omega) s(0), \quad a_1 = \frac{1}{4}\int_{\partial\Omega}\int_{\mathbf{R}^n} |z\cdot n_x| s(z) s(-z)\, dz\, dA. \tag{1.2}$$

Here n_x is the inward pointing unit normal at $x \in \partial\Omega$ and dA is surface measure on $\partial\Omega$. In [**7**] Widom found a generalization of this in the case k is matrix valued and $L_2(\alpha\Omega)$ consists of vector valued functions.

In this paper the following extension is considered. If F is a function analytic on the spectrum of T_α, then $F(T_\alpha)$ is defined and will be trace class if $F(0) = 0$. It is shown here that in the scalar case there is an expansion

$$\operatorname{tr} F(T_\alpha) = a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + o(\alpha^{n-2}), \qquad \alpha \to \infty,$$

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where a_0 and a_1 are given by (1.2) in the case F is the logarithm and the evaluation of a_2 is new. In the matrix case an expression is obtained only when F is a power, i.e., $F(\lambda) = \lambda^m$ for some positive integer m. The coefficients a_0 and a_1 agree with those given in [7] and again the evaluation of a_2 is new.

The assumptions and notation will be as follows. Let $k \in L_1(\mathbf{R}^n)$ and suppose $\sigma = \hat{k}$ is also in $L_1(\mathbf{R}^n)$. In addition, it is assumed that

$$\int_{\mathbf{R}^n} |x|^2 |k(x)|\, dx < \infty, \tag{1.3}$$

where, if k is matrix valued, $|k(x)|$ denotes the Hilbert-Schmidt norm of $k(x)$.

Ω will be a compact subset of $\mathbf{R}^n$ $(n > 1)$ whose boundary is a C^3 hypersurface. $T^*(\partial\Omega)$ will denote the cotangent bundle of $\partial\Omega$ which is identified with the tangent bundle $T(\partial\Omega)$ via the Riemannian metric on $\partial\Omega$ ($\mathbf{R}^n$ inner product restricted to tangent spaces). This induces a natural measure $d\bar{\xi}\, dA$ on $T^*(\partial\Omega)$, where dA is surface measure on $\partial\Omega$ and $d\bar{\xi}$ is Lebesgue measure on the hyperplane $T_x(\partial\Omega)$. The notation $\xi = (\bar{\xi}, \eta) = (\xi^1, \ldots, \xi^{n-1}, \eta)$ will be used for vectors in $T_x(\partial\Omega) \times \mathbf{R}$, the last coordinate η being with respect to the unit inward normal to $\partial\Omega$ at x. Superscripts $i, j = 1, \ldots, n-1$ on a function will denote differentiation in the tangential direction (i.e., $\tau^i = \partial\tau/\partial\bar{\xi}^i$) and $\operatorname{grad}_{\bar{\xi}}(\tau)$ will mean $(\tau^1, \ldots, \tau^{n-1})$. The superscript n will denote differentiation in the normal direction.

Finally, L will denote the second fundamental form for $\partial\Omega$ with respect to the inward unit normal and H will denote $n-1$ times the mean curvature of $\partial\Omega$. The main result in the scalar case can now be stated.

THEOREM 1.1. *Let F be analytic on a disc of radius greater than $\|k\|_1$ and suppose $F(0) = 0$. Then under the above assumptions*

$$\operatorname{tr} F(T_\alpha) = a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + o(\alpha^{n-2}), \qquad \alpha \to \infty,$$

where

$$a_0 = \operatorname{vol}(\Omega)\frac{1}{(2\pi)^n}\int F(\sigma(\xi))\, d\xi,$$

$$a_1 = \frac{1}{4}\frac{1}{(2\pi)^n}\int_{T^*(\partial\Omega)\times\mathbf{R}}\int \frac{F(\sigma(\eta)) - F(\sigma(\zeta))}{\sigma(\eta) - \sigma(\zeta)}\,\frac{\sigma^n(\zeta) - \sigma^n(\eta)}{\zeta - \eta}\, d\zeta\, d\xi\, dA,$$

$$a_2 = -\frac{1}{8\pi^2}\frac{1}{(2\pi)^n}\int_{T^*(\partial\Omega)\times\mathbf{R}}\int\int\left\{\sum_{a\in A_3}\frac{F(\sigma(\eta^a))}{(\sigma(\eta^a) - \sigma(\zeta_1^a))(\sigma(\eta^a) - \sigma(\zeta_2^a))}\right\}$$

$$\times\left\{L\left(\operatorname{grad}_{\bar{\xi}}\sigma(\zeta_1), \operatorname{grad}_{\bar{\xi}}\sigma(\zeta_2)\right) - H\cdot\sigma^n(\zeta_1)\sigma^n(\zeta_2)\right\}\frac{d\zeta_1}{\zeta_1 - \eta}\,\frac{d\zeta_2}{\zeta_2 - \eta}\, d\xi\, dA.$$

Here A_3 is the alternating group on the 3 *symbols η, ζ_1, ζ_2, and expressions like $\sigma(\zeta_1)$ mean $\sigma(\xi^1, \ldots, \xi^{n-1}, \zeta_1)$.*

The proof proceeds as follows. The kernel of T_α^m at a point $(x, y) \in \alpha\Omega \times \alpha\Omega$ is

$$\int \cdots \int k(x - x_1)k(x_1 - x_2)\cdots k(x_{m-2} - x_{m-1})k(x_{m-1} - y)$$
$$\times \chi_{\alpha\Omega}(x_1)\cdots\chi_{\alpha\Omega}(x_{m-1})\, dx_1 \cdots dx_{m-1},$$

where the integration is taken over $\mathbf{R}^n \times \cdots \times \mathbf{R}^n$ and where χ_S denotes the characteristic function of the set S. Change variables: let $\bar{x}_1 = x - x_1$, $\bar{x}_2 = x_1 - x_2, \ldots, \bar{x}_{m-1} = x_{m-2} - x_{m-1}$. This gives

$$\int \cdots \int k(x_1) \cdots k(x_{m-1}) k(x - y - x_1 - \cdots - x_{m-1}) \chi_{\alpha\Omega}(x - x_1)$$
$$\times \chi_{\alpha\Omega}(x - x_1 - x_2) \cdots \chi_{\alpha\Omega}(x - x_1 - \cdots - x_{m-1}) \, dx_1 \cdots dx_{m-1}.$$

Under the above hypotheses this expression is continuous for $m \geqslant 2$ (the case $m = 1$ is easily handled; see [**3**, pp. 506–507]) so to find $\operatorname{tr} T_\alpha^m$ the trace of the kernel is integrated over the diagonal of $\alpha\Omega \times \alpha\Omega$. The result is

(1.4)
$$\operatorname{tr}(T_\alpha^m) = \int \cdots \int \operatorname{tr} k(x_1) \cdots k(x_{m-1}) k(-x_1 - \cdots - x_{m-1})$$
$$\times \operatorname{vol}(\alpha\Omega \cap (\alpha\Omega + x_1) \cap \cdots \cap (\alpha\Omega + x_1 + \cdots + x_{m-1})) \, dx_1 \cdots dx_{m-1}.$$

In §2 an expression is obtained for the volume in the integrand and in §3 this is used to derive a formula for $\operatorname{tr}(T_\alpha^m)$. The latter and an identity of F. Spitzer (see [**1**]) are then used in §4 to prove Theorem 1.1.

If $\|k\|_1 < 1$ then $\log(1 + \lambda)$ satisfies the hypotheses of Theorem 1.1 and there results an expansion for $\operatorname{tr} \log(1 + T_\alpha)$. It is not immediate that the a_1 term of this expansion is the same as the a_1 term appearing in (1.2). In §5 this is shown to be the case. An argument similar to but more complicated than the one in §5 also shows that the expression for a_2 in Theorem 1.1 is equivalent, in the case $F(\lambda) = \log(1 + \lambda)$, to one recently derived by Widom [**8**].

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2. The volume estimate. In obtaining an expression for the volume in the integrand of (1.4) it is convenient to divide by α^n and set $\varepsilon = \alpha^{-1}$.

THEOREM 2.1. *Let Ω be a compact subset of $\mathbf{R}^n$ $(n > 1)$ whose boundary is a C^3 hypersurface. Let $v_1, \ldots, v_r \in \mathbf{R}^n$ and define $\Omega_\varepsilon = \Omega \cap (\Omega + \varepsilon v_1) \cap \cdots \cap (\Omega + \varepsilon v_r)$. Then*

$$\operatorname{vol}(\Omega \setminus \Omega_\varepsilon) = \int_{\partial\Omega} \max_{i=1,\ldots,r} \left\{0, \varepsilon v_i \cdot n_x + \frac{\varepsilon^2}{2} L(v_{i_t}, v_{i_t})\right\} dA(x)$$
$$- \frac{\varepsilon^2}{2} \int_{\partial\Omega} \left(\max_{i=1,\ldots,r} \{0, v_i \cdot n_x\}\right)^2 H \, dA(x) + o(\varepsilon^2), \qquad \varepsilon \to 0,$$

where n_x is the unit inward normal at $x \in \partial\Omega$, L and H are as in the introduction, and w_t is the component of w tangent to $\partial\Omega$ at x.

PROOF. Since $\partial\Omega$ is compact there exists an ε_0 and an ε_0-tubular neighborhood N_0 of $\partial\Omega$ such that each $x \in N_0$ can be written uniquely as $x = \bar{x} + sn_{\bar{x}}$, where $\bar{x} \in \partial\Omega$ and $|s| < \varepsilon_0$. If ε is small enough then $\Omega \setminus \Omega_\varepsilon \subset N_0$.

Let $\{C_\tau, \psi_\tau\}$ be an atlas of coordinate neighborhoods covering $\partial\Omega$ and pick a finite subcover. Let $D_\tau = \{x \in N_0$: if $x = \bar{x} + sn_{\bar{x}}$, then $\bar{x} \in C_\tau\}$. Define ϕ_τ: $D_\tau \to \mathbf{R}^{n-1} \times \mathbf{R}$ as follows: if $x = \bar{x} + sn_{\bar{x}} \in D_\tau$ and $\psi_\tau(\bar{x}) = \bar{u} \in \mathbf{R}^{n-1}$, then ϕ_τ: $\bar{x} + sn_{\bar{x}} \to \bar{u} + s\nu$, where ν is the unit vector in $\mathbf{R}^{n-1} \times \mathbf{R}$ normal to $\mathbf{R}^{n-1}$. By the compactness of Ω there exist open sets $N_\tau \subset D_\tau$ such that the N_τ are an open cover of N_0 and the distance from N_τ to the complement of D_τ is, for all τ, greater than some ε_1. If ε is chosen such that $\max|\varepsilon v_i| < \varepsilon_1$ then, for all τ, $x \in N_\tau$ implies $x - \varepsilon v_i \in D_\tau$. Let $B_\tau = N_\tau \cap \partial\Omega \subset C_\tau$.

Let $\{\rho_\tau\}$ be a partition of unity subordinate to the cover $\bigcup B_\tau$ of $\partial\Omega$. Each ρ_τ extends to a function $\tilde{\rho}_\tau$ on $N_\tau \cap \Omega \setminus \Omega_\varepsilon$ by defining $\tilde{\rho}_\tau(\bar{x} + sn_{\bar{x}}) = \rho_\tau(\bar{x})$. It now suffices to prove

$$\int_{N_\tau \cap \Omega \setminus \Omega_\varepsilon} \tilde{\rho}_\tau \, dV = \int_{B_\tau} \rho_\tau(\bar{x}) \max_i \left\{0, \varepsilon v_i \cdot n_{\bar{x}} + \frac{\varepsilon^2}{2} L(v_{i_t}, v_{i_t})\right\} dA - \frac{\varepsilon^2}{2} \int_{B_\tau} \rho_\tau(\bar{x}) \Big(\max_i \{0, v_i \cdot n_{\bar{x}}\}\Big)^2 H \, dA + o(\varepsilon^2), \tag{2.1}$$

where dV denotes the volume element on $\mathbf{R}^n$.

In what follows the index τ will be dropped. From the construction of ϕ it follows that for $y \in D$, $y \in \Omega \cap D$ if and only if $\phi(y) \cdot \nu = s \geqslant 0$. Hence for $x \in N$,

$$x \in \Omega_\varepsilon \cap N = N \cap \Omega \cap (\Omega + \varepsilon v_1) \cap \cdots \cap (\Omega + \varepsilon v_r)$$

if and only if $s \geqslant 0$ and, for all i, $\phi(x - \varepsilon v_i) \cdot \nu \geqslant 0$. By Taylor's theorem $x \in N \cap \Omega_\varepsilon$ if and only if $s \geqslant 0$ and, for all i,

$$\phi(x) \cdot \nu - \varepsilon d\phi_x(v_i) \cdot \nu + \frac{\varepsilon^2}{2} d^2\phi_x(v_i, v_i) \cdot \nu + R(\varepsilon) \geqslant 0, \tag{2.2}$$

where $R(\varepsilon) = o(\varepsilon^2)$, the estimate being uniform over x since $d^2\phi$ is continuous.

LEMMA 2.2. *For all* $w \in \mathbf{R}^n$ *and all* $x = \bar{x} + sn_{\bar{x}} \in N$
(i) $d\phi_x(w) \cdot \nu = w \cdot n_{\bar{x}}$,
(ii) $d^2\phi_x(w, w) \cdot \nu = -L(w_t, w_t) + r(s)$,
where $\lim_{s \to 0} r(s) = 0$ *uniformly in* x *and* L *is evaluated at* $\bar{x}$.

The proof will appear later in the section. From (2.2) and the first part of the lemma, $x \in N \cap \Omega \setminus \Omega_\varepsilon$ if and only if

$$s \geqslant 0 \text{ and, for some } i, \quad s - \varepsilon v_i \cdot n_{\bar{x}} + \frac{\varepsilon^2}{2} d^2\phi_x(v_i, v_i) \cdot \nu + R(\varepsilon) < 0. \tag{2.3}$$

By part (ii) of the lemma, $x \in N \cap \Omega \setminus \Omega_\varepsilon$ if and only if $s \geqslant 0$ and, for some i,

$$s - \varepsilon v_i \cdot n_{\bar{x}} - \frac{\varepsilon^2}{2} L(v_{i_t}, v_{i_t}) + R_1(\varepsilon) < 0,$$

where $R_1(\varepsilon) = \varepsilon^2 r(s)/2 + R(\varepsilon) = o(\varepsilon^2)$ (since (2.3) implies $s \to 0$ as $\varepsilon \to 0$). It follows that

$$N \cap \Omega \setminus \Omega_\varepsilon = N \cap \Omega \setminus S_\varepsilon, \tag{2.4}$$

where $S_\varepsilon = \{x \in N\colon s \geqslant 0$ and, for all i, $s - \varepsilon v_i \cdot n_{\bar{x}} - \varepsilon^2 L(v_{i_t}, v_{i_t})/2 + R(\varepsilon) \geqslant 0\}$. Define $I_0 = \{x \in N\colon s \geqslant 0$ and, for all i, $s - \varepsilon v_i \cdot n_{\bar{x}} - \varepsilon^2 L(v_{i_t}, v_{i_t})/2 \geqslant 0\}$.

LEMMA 2.3. $\operatorname{vol}(S_\varepsilon \triangle I_0) = o(\varepsilon^2)$, $\varepsilon \to 0$.

Again the proof is deferred to later in the section. From (2.4) and Lemma 2.3 it follows that $N \cap \Omega \setminus \Omega_\varepsilon$ can be replaced by $N \cap \Omega \setminus I_0$ in (2.1).

By the change of variables formula,

(2.5)
$$\int_{\Omega \cap N \setminus I_0} \tilde{\rho}\, dV = \int_{\phi(\Omega \cap N \setminus I_0)} (\tilde{\rho} \circ \phi^{-1}) |d\phi^{-1}{}_u| du^1 \cdots du^{n-1}\, ds$$
$$= \int_{\phi(\Omega \cap N \setminus I_0)} (\tilde{\rho} \circ \phi^{-1}) \Big\{ |d\phi^{-1}{}_{\bar{u}}| + d\Big(|d\phi^{-1}{}_{(\cdot)}|\Big)_{\bar{u}}(s\nu) + o(s) \Big\}\, du^1 \cdots du^{n-1}\, ds,$$

where by the definitions of ϕ and I_0

$$\phi(\Omega \cap N \setminus I_0) = \left\{ u \in \phi(N)\colon 0 \leqslant s < \max_i \left(0, \varepsilon v_i \cdot n_{\bar{x}} + \frac{\varepsilon^2}{2} L(v_{i_t}, v_{i_t})\right)\right\}. \tag{2.6}$$

Integrating the $o(s)$ term within the indicated limits yields something which is $o(\varepsilon^2)$. Since $\tilde{\rho}$ is constant with respect to s, integrating the first term of (2.5) with respect to s yields

$$\int_{\phi(B)} (\rho \circ \phi^{-1}) \max_i \left\{ 0, \varepsilon v_i \cdot n_{\bar{x}} + \frac{\varepsilon^2}{2} L(v_{i_t}, v_{i_t}) \right\} |d\phi^{-1}{}_{\bar{u}}| du^1 \cdots du^{n-1}$$

which equals the first term on the right side of (2.1).

Integrating the remaining term in (2.5) with respect to s yields

$$\frac{\varepsilon^2}{2} \int_{\phi(B)} (\rho \circ \phi^{-1}) (\max(0, v_i \cdot n_{\bar{x}}))^2 d\Big(|d\phi^{-1}{}_{(\cdot)}|\Big)_{\bar{u}}(\nu)\, du^1 \cdots du^{n-1} + o(\varepsilon^2).$$

By a classical result from surface theory [**5**, p. 159] this last integral equals

$$-\frac{\varepsilon^2}{2} \int_B \rho (\max(0, v_i \cdot n_x))^2 H\, dA.$$

Comparison with (2.1) shows the theorem is proved.

COROLLARY 2.4. *Except for a set of* $(v_1, \ldots, v_r)$ *of measure zero in* $\mathbf{R}^n \times \cdots \times \mathbf{R}^n$

$$\operatorname{vol}(\Omega \setminus \Omega_\varepsilon) = \varepsilon \int_{\partial\Omega} \max_{i = 1, \ldots, r} \{0, v_i \cdot n_x\}\, dA(x)$$
$$+ \frac{\varepsilon^2}{2} \sum_{q=1}^{r} \int_{S_q} L(v_{q_t}, v_{q_t}) - (v_q \cdot n_x)^2 H\, dA(x) + o(\varepsilon^2), \qquad \varepsilon \to 0,$$

where $S_q = \{x \in \partial\Omega\colon v_q \cdot n_x = \max_i(0, v_i \cdot n_x)\}$.

PROOF. For fixed $x \in \partial\Omega$ and fixed $v \in \mathbf{R}^n$, the set of $w \in \mathbf{R}^n$ such that $v \cdot n_x = w \cdot n_x$ is a hyperplane and so has measure zero in $\mathbf{R}^n$. Thus the set of $(v_1, v_2) \in \mathbf{R}^n \times \mathbf{R}^n$

such that $v_1 \cdot n_x = v_2 \cdot n_x$ has measure zero and it follows that the set of $(v_1, \ldots, v_r) \in \mathbf{R}^n \times \cdots \times \mathbf{R}^n$ such that $\max_i(0, v_i \cdot n_x)$ occurs at more than one of the v_i has measure zero in $\mathbf{R}^n \times \cdots \times \mathbf{R}^n$.

Thus the set $E = \{(x, v_1, \ldots, v_r) \in \partial\Omega \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n$: $\max(0, v_i \cdot n_x)$ occurs at more than one of the $v_i\}$ has measure zero in $\partial\Omega \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n$. By Fubini's theorem, for almost every $(v_1, \ldots, v_r) \in \mathbf{R}^n \times \cdots \times \mathbf{R}^n$ the set $E_{(v_1, \ldots, v_r)} = \{x \in \partial\Omega$: $(x, v_1, \ldots, v_r) \in E\}$ has measure zero in $\partial\Omega$. Thus for almost every $(v_1, \ldots, v_r)$ the sets $S_q = \{x \in \partial\Omega$: $v_q \cdot n_x > \max_{q \neq 1}(0, v_i \cdot n_x)\}$, $q = 1, \ldots, r$, and $S_0 = \{x \in \partial\Omega$: $\max_i(0, v_i \cdot n_x) = 0\}$ form a disjoint cover of $\partial\Omega$ except for a set of measure zero. Note that on S_q, $q = 1, \ldots, r$, the error obtained by replacing the integrand

$$\max_i \left(0, \varepsilon v_i \cdot n_x + \frac{\varepsilon^2}{2} L(v_{i_t}, v_{i_t})\right)$$

in Theorem 2.1 by $\varepsilon v_q \cdot n_x + \varepsilon^2 L(v_{q_t}, v_{q_t})/2$ is $O(\varepsilon^2)$ and likewise for replacing the integrand with 0 on S_0.

For $\delta > 0$, define $S_{q\delta} = \{x \in \partial\Omega$: $v_q \cdot n_x > \max_{i \neq q}(0, v_i \cdot n_x) + \delta\}$ for $q = 1, \ldots, r$ and $S_{0\delta} = \{x \in \partial\Omega$: $\max_i(v_i \cdot n_x) < -\delta\}$. Then $S_{q\delta} \subset S_q$ for $q = 0, 1, \ldots, r$ and on each $S_{q\delta}$ there is an $\varepsilon_{q\delta}$ such that for $\varepsilon < \varepsilon_{q\delta}$ no error results from the replacements indicated above.

Hence for $\varepsilon < \varepsilon_{q\delta}$ the total error on S_q, $q = 0, 1, \ldots, r$, is less than

$$D_q \cdot \text{meas}\,(S_q \setminus S_{q\delta})\varepsilon^2$$

for some constant D_q. Since meas $(S_q \setminus S_{q\delta})$ can be made arbitrarily small by choosing δ small enough, the error is $o(\varepsilon^2)$.

The remainder of the section consists of the proofs of Lemmas 2.2 and 2.3 followed by another lemma which will be useful later.

PROOF OF LEMMA 2.2. To prove part (i) note that since

$$d\phi_x(w) \cdot \nu = d\phi_x(w_t) \cdot \nu + d\phi_x((w \cdot n_{\bar{x}})n_{\bar{x}}) \cdot \nu$$

it suffices to prove

$$d\phi_x(n_{\bar{x}}) = \nu \quad \text{and} \quad d\phi_x(w_t) \cdot \nu = 0. \tag{2.7}$$

For each fixed s_0 the map ϕ^{-1}: $(\bar{u}, s_0) \to \bar{x}(\bar{u}) + s_0 n_{\bar{x}}(\bar{u})$ describes the hypersurface $B + s_0 n_{\bar{x}}$. The vectors $\partial(\bar{x} + s_0 n_{\bar{x}})/\partial u^i|_{\bar{u}_0}$ form a basis for the tangent space to $B + s_0 n_{\bar{x}}$ at $\bar{x}(\bar{u}_0) + s_0 n_{\bar{x}}(\bar{u}_0)$. Thus $d\phi^{-1}{}_{(\bar{u}_0, s_0)}$ sends vectors $(\bar{u}, 0)$ to vectors tangent to $B + s_0 n_{\bar{x}}$ at $\bar{x}(\bar{u}_0) + s_0 n_{\bar{x}}(\bar{u}_0)$ and sends $\nu = (0, \ldots, 0, 1)$ to $n_{\bar{x}}$. Hence it suffices to show that for any $x = \bar{x} + s n_{\bar{x}}$, $T_x(B + s n_{\bar{x}}) = T_{\bar{x}}(\partial\Omega)$. But

$$n_{\bar{x}} \cdot \frac{\partial}{\partial u^i}(\bar{x} + s n_{\bar{x}}) = s n_{\bar{x}} \cdot \frac{\partial}{\partial u^i} n_{\bar{x}} = \frac{1}{2} s \frac{\partial}{\partial u^i}(n_{\bar{x}} \cdot n_{\bar{x}}) = 0,$$

so the tangent space $T_x(B + s n_{\bar{x}})$ is normal to $n_{\bar{x}}$ and hence the same as $T_{\bar{x}}(\partial\Omega)$.

Now the proof of part (ii). Since $d^2\phi$ is uniformly continuous on N,

$$d^2\phi_x(w, w) \cdot \nu = d^2\phi_{\bar{x}}(w, w) \cdot \nu + r(s)$$

where $\lim_{s\to 0} r(s) = 0$ uniformly in x. Also, since

$$d^2\phi_x(w, n_{\bar{x}}) \cdot \nu = \left(\lim_{s\to 0} \frac{d\phi_{x+sn_{\bar{x}}}(w) - d\phi_x(w)}{s}\right) \cdot \nu$$

$$= \lim_{s\to 0} \frac{w \cdot n_{\bar{x}} - w \cdot n_{\bar{x}}}{s} \quad \text{(by Lemma 2.2)}$$

$$= 0,$$

$$d^2\phi_{\bar{x}}(w, w) \cdot \nu = d^2\phi_{\bar{x}}(w_t, w_t) \cdot \nu. \tag{2.8}$$

Hence it suffices to show

$$d^2\phi_{\bar{x}}(w_t, w_t) \cdot \nu = -L(w_t, w_t). \tag{2.9}$$

Fix $\bar{x} \in B$. It suffices to assume $\bar{x} = 0 \in \mathbf{R}^n$ and that ψ projects B orthogonally onto a neighborhood of the origin in $T_0(\partial\Omega)$ (every $\bar{x} \in \partial\Omega$ has a neighborhood $B_{\bar{x}}$ which is diffeomorphic to a neighborhood of the origin in $T_{\bar{x}}(\partial\Omega)$ via orthogonal projection; pick a finite subcover and proceed as before).

For all $h \in T_0(\partial\Omega)$ with h sufficiently small, h is in the image of ψ and one can define $k_h = \bar{x}(h)$. By Taylor's theorem

$$k_h = \bar{x}(h) = \bar{x}(0) + (d\bar{x})_0(h) + \frac{1}{2}\sum_{i,j=1}^{n-1} \frac{\partial^2 \bar{x}}{\partial u^j \partial u^i}(0)h^i h^j + o\left(|h|^2\right).$$

Since $\bar{x}(0) = 0$ and $(d\bar{x})_0(h) \in T_0(\partial\Omega)$ it follows that

$$k_h \cdot n_0 = \tfrac{1}{2}L(h, h) + o(|h|)^2. \tag{2.10}$$

On the other hand, Taylor's theorem and (2.7) give

$$\phi(k_h) - d\phi_0((k_h)_t) - (k_h \cdot n_0) \cdot \nu = \tfrac{1}{2}d^2\phi_0(k_h, k_h) + o\left(|k_h|^2\right).$$

Since $k_h = \bar{x}(h) \in \partial\Omega$ and $\phi|_{\partial\Omega} = \psi$ the first two terms on the left side are orthogonal to ν. Thus

$$-k_n \cdot n_0 = \tfrac{1}{2}d^2\phi_0((k_h)_t, (k_h)_t) \cdot \nu + o\left(|k_h|^2\right)$$

$$= \tfrac{1}{2}d^2\phi_0(h, h) \cdot \nu + o\left(|h|^2\right)$$

(since $(k_h)_t = h$, any $o(|k_h|^2)$ term is $o(|h|^2)$). Comparing this with (2.10) yields (2.9) and hence the result.

Proof of Lemma 2.3. For any real number δ define $I_\delta = \{x \in N\colon s \geqslant 0 \text{ and, for all } i,\ s - \varepsilon v_i \cdot n_{\bar{x}} - \varepsilon^2 L(v_{i_t}, v_{i_t})/2 \geqslant \delta\varepsilon^2\}$. It is easy to show that for each $\delta > 0$ there exists an ε_δ such that

$$\varepsilon < \varepsilon_\delta \quad \text{implies} \quad I_\delta \subset S_\delta \subset I_{-\delta}. \tag{2.11}$$

Next,

$$\varepsilon < \varepsilon_\delta \quad \text{implies} \quad (S_\varepsilon \triangle I_0) \subset I_{-\delta} \setminus I_\delta. \tag{2.12}$$

For, $x \in S_\varepsilon \triangle I_0$ if and only if $x \in S_\varepsilon$ but $x \notin I_0$ or $x \notin S_\varepsilon$ but $x \in I_0$. In the first case, $x \in S_\varepsilon$ implies $x \in I_{-\delta}$ by (2.11) and $x \notin I_0$ implies $x \notin I_\delta$. In the second case, $x \notin S_\varepsilon$ implies $x \notin I_\delta$ by (2.11) and $x \in I_0$ implies $x \in I_{-\delta}$.

From (2.12) it follows that for $\varepsilon < \varepsilon_\delta$,

$$\mathrm{vol}(S_\varepsilon \triangle I_0) \leqslant \mathrm{vol}(I_{-\delta} \setminus I_\delta) = \int_{\phi(I_{-\delta}\setminus I_\delta)} |d\phi^{-1}{}_{(\bar{u},s)}| du^1 \cdots du^{n-1}\, ds.$$

But

$$\phi(I_{-\delta} \setminus I_\delta) = \left\{ u \in \phi(N) : s \geqslant 0 \text{ and } -\delta\varepsilon^2 + \max\left(\varepsilon v_i \cdot n_{\bar{x}} + \frac{\varepsilon^2}{2} L(v_{i_t}, v_{i_t}) \right) \right.$$

$$\left. \leqslant s < \delta\varepsilon + \max_i \left(\varepsilon v_i \cdot n_{\bar{x}} + \frac{\varepsilon^2}{2} L(v_{i_t}, v_{i_t}) \right) \right\}.$$

Thus the above integral is

$$\leqslant 2\delta\varepsilon^2 \sup_{\phi(N)} |d\phi^{-1}{}_{(\cdot)}| \int_{\phi(\partial\Omega \cap I_{-\delta}\setminus I_\delta)} du^1 \cdots du^{n-1}$$

$$\leqslant 2\delta\varepsilon^2 M,$$

where $M = \sup_{\phi(N)} |d\phi^{-1}{}_{(\cdot)}| \mathrm{vol}_{\mathbf{R}^{n-1}}(\phi(B))$ (by the compactness of Ω one can guarantee this is finite). Thus it is established that for any $\delta > 0$ there is an ε_δ such that for all $\varepsilon < \varepsilon_\delta$, $\mathrm{vol}(S_\varepsilon \triangle I_0)/\varepsilon^2 \leqslant 2\delta M$.

LEMMA 2.5. *Let $\{v_1, \ldots, v_r\}$ be any finite set in $\mathbf{R}^n$ and define $\Omega_v = \Omega \cap (\Omega + v_1) \cap \cdots \cap (\Omega + v_r)$. There exists a constant C, independent of the v_i and of r, such that*

$$\left| \mathrm{vol}(\Omega \setminus \Omega_v) - \int_{\partial\Omega} \max_i (0, v_i \cdot n_{\bar{x}})\, dA \right| \leqslant C \max_i \left(|v_i|^2 \right).$$

PROOF. For any positive number p it can be assumed that $\max_i(|v_i|) < p$. For,

$$\left| \mathrm{vol}(\Omega \setminus \Omega_v) - \int_{\partial\Omega} \max_i (0, v_i \cdot n_{\bar{x}})\, dA \right| \leqslant \mathrm{vol}(\Omega) + \max_i \left(|v_i| \right) \mathrm{vol}(\partial\Omega)$$

and if $\max(|v_i|) \geqslant p$ then the right side is less than $C_1 \max(|v_i|)^2$ where

$$C_1 = \frac{\mathrm{vol}(\Omega) + p\,\mathrm{vol}(\partial\Omega)}{p^2}.$$

Take $\max(|v_i|)$ small enough so that $\Omega \setminus \Omega_v \subset N_0$ and so that $x - v_i \in \mathrm{domain}\, \phi_\tau$ whenever $x \subset N_\tau$. Again using a partition of unity argument, it is enough to show

$$\left| \int_{N_\tau \cap \Omega \setminus \Omega_v} \tilde{\rho}_\tau\, dV - \int_{B_\tau} \rho_\tau(\bar{x}) \max_i (0, v_i \cdot n_{\bar{x}})\, dA \right| \leqslant C_\tau \max_i \left(|v_i|^2 \right).$$

As before, the index τ will be dropped.

$x \in \Omega_v \cap N$ if and only if $s \geqslant 0$ and, for all i, $\phi(x - v_i) \cdot \nu \geqslant 0$. By Taylor's theorem $x \in \Omega_v \cap N$ if and only if $s \geqslant 0$ and, for all i, $\phi(x) \cdot \nu - d\phi_x(v_i) \cdot \nu + R(v_i) \geqslant 0$ where $|R(v_i)| \leqslant M|v_i|^2$, the bound being uniform over x since $d^2\phi$ is bounded on N. By Lemma 2.2, $x \in \Omega_v \cap N$ if and only if $s \geqslant 0$ and, for all i, $s - v_i \cdot n_{\bar{x}} + R(v_i) \geqslant 0$.

Define $J_M = \{x \in N\colon\ s \geqslant 0$ and, for all i, $s - v_i \cdot n_{\bar{x}} \geqslant M(\max|v_i|^2)\}$. By an argument identical to that of Lemma 2.3 there is an M_1 such that $\operatorname{vol}((\Omega_v \cap N)\triangle J_0) \leqslant M_1 \max(|v_i|^2)$ so $N \cap \Omega \setminus \Omega_v$ may be replaced by $N \cap \Omega \setminus J_0$.

By the mean value theorem (writing $u = \bar{u} + s$)

$$\int_{N\cap\Omega\setminus J_0} \tilde{\rho}_\tau\, dV = \int_{\phi(\Omega\cap N\setminus J_0)} (\tilde{\rho}\circ\phi^{-1})|d\phi^{-1}{}_u|du^1 \cdots du^{n-1}\, ds$$

$$= \int_{\phi(\Omega\cap N\setminus J_0)} (\tilde{\rho}\circ\phi^{-1})\Big\{|d\phi^{-1}{}_{\bar{u}}| + d\big(|d\phi^{-1}{}_{(\cdot)}|\big)_{\bar{u}+s'v}(s\nu)\Big\}\, du^1 \cdots du^{n-1}\, ds,$$

where $0 \leqslant s' \leqslant s$ and $\phi(\Omega \cap N\setminus J_0) = \{u \in \phi(N)\colon 0 \leqslant s < \max_i(0, v_i \cdot n_{\bar{x}})\}$. Performing the first integral on the right with respect to s (keep in mind that $\tilde{\rho}$ is independent of s) and subtracting yields

$$\left|\int_{N\cap\Omega\setminus J_0} \tilde{\rho}\, dV - \int_B \rho(\bar{x}) \max_i (0, v_i\cdot n_{\bar{x}})\, dA\right|$$

$$= \left|\int_{\phi(\Omega\cap N\setminus J_0)} s(\tilde{\rho}\circ\phi^{-1})\big(d|d\phi^{-1}{}_{(\cdot)}|\big)_{\bar{u}+s'\nu}(\nu)\, du^1 \cdots du^{n-1}\, ds.\right|$$

If $C_1 = \sup\{|d|d\phi^{-1}{}_{(\cdot)}|| \cdot |\tilde{\rho}\circ\phi^{-1}|\}$ the last expression is

$$\leqslant C_1 \int_{\phi(\Omega\cap N\setminus J_0)} s\, du^1 \cdots du^{n-1}\, ds$$

$$= \frac{1}{2} C_1 \int_{\phi(B)} \Big(\max_i (0, v_i \cdot n_{\bar{x}})\Big)^2\, du^1 \cdots du^{n-1}$$

$$\leqslant \max\big(|v_i|^2\big) \cdot \frac{1}{2} C_1 \int_{\phi(B)} du^1 \cdots du^{n-1}.$$

3. $\operatorname{tr}(T_\alpha^m)$ **in the matrix case.** In this section k will be matrix valued. The notation $f * g$ will denote the convolution of f and g:

$$f * g(x) = \int_{\mathbf{R}^n} f(x-y)g(y)\, dy.$$

Note that (1.3) together with the assumption $\sigma \in L_1$ implies

(A) $\int|x|\,|k(x)|\, dx < \infty$ which in turn implies $\int|x|\,|k(x)|^2\, dx < \infty$, so the hypotheses of [7] are satisfied;

(B) $\int|x|^2|k(x)|^2\, dx < \infty$.

The following two lemmas will be needed.

LEMMA 3.1. *With* $|k|^{(r)}$ *denoting the r-fold convolution of* $|k|$ *with itself*,

(i) $$\int|x|\big(|k|^{(m-1)}(x)\big)^2\, dx \leqslant (m-1)^2 \int|x|\,|k(x)|^2\, dx \left\{\int|k(x)|dx\right\}^{2(m-2)};$$

(ii) *the same estimate holds with* $|x|$ *replaced by* $|x|^2$ *on both sides.*

The proof is deferred to the end of the section. An easy consequence of Lemma 3.1 is

LEMMA 3.2. (i)

$$\int \cdots \int |x|\,|k(x)|\,|k(x_1)| \cdots |k(x_{m-2})|\,|k(-x-x_1-\cdots-x_{m-2})|dx_1 \cdots dx_{m-2}\,dx$$
$$\leqslant (m-1)\int |x|\,|k(x)|^2\,dx\left\{\int |k(x)|dx\right\}^{m-2}.$$

(ii) *The same estimate holds with* $|x|$ *replaced by* $|x|^2$ *on both sides.*

The statement of the main theorem of this section requires the following notation. The Hilbert transform (with respect to η) of an $L_2(T_x(\partial\Omega)\times\mathbf{R})$ function $\sigma(\xi^1,\ldots,\xi^{n-1},\eta)$ is

$$\tilde{\sigma}(\eta) = [i\,\mathrm{sgn}(t)\sigma(t)]^{\vee},$$

the inverse Fourier transform being taken with respect to the last variable. It is convenient also to write

$$\tilde{\sigma}(\eta) = \frac{1}{\pi}\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{\sigma(\xi^1,\ldots,\xi^{n-1},\zeta)}{\zeta-\eta}d\zeta$$

(the expression

$$\frac{1}{\pi}\int_{|\zeta-\eta|>\varepsilon}\frac{\sigma(\xi^1,\ldots,\xi^{n-1},\zeta)}{\zeta-\eta}d\zeta$$

converges to $\tilde{\sigma}(\eta)$ in the L_2 norm as ε tends to zero). At each $x\in\partial\Omega$ define

(3.1) $$\sigma_{\pm} = \tfrac{1}{2}(\sigma \pm i\tilde{\sigma})$$

and then, inductively,

$$\Pi^1_+(\sigma) = \sigma_+, \qquad \Pi^1_-(\sigma) = \sigma_-,$$
$$\Pi^n_+(\sigma) = \left(\sigma\Pi^{n-1}_+(\sigma)\right)_+, \qquad \Pi^n_-(\sigma) = \left(\Pi^{n-1}_-(\sigma)\cdot\sigma\right)_-.$$

THEOREM 3.3. *Under the assumptions outlined in the introduction*

$$\mathrm{tr}(T^m_\alpha) = a_0\alpha^n + a_1\alpha^{n-1} + a_2\alpha^{n-2} + o(\alpha^{n-2}), \qquad \alpha\to\infty,$$

where

$$a_0 = \frac{1}{(2\pi)^n}\mathrm{vol}(\Omega)\int \mathrm{tr}\,\sigma^m(\xi)\,d\xi,$$

$$a_1 = \frac{-i}{(2\pi)^n}\sum_{q=1}^{m-1}\int_{T^*(\partial\Omega)\times\mathbf{R}}\mathrm{tr}\,\Pi^q_+(\sigma)''(\xi)\Pi^{m-q}_-(\sigma)(\xi)\,d\xi\,dA$$

and

$$a_2 = -\frac{1}{2(2\pi)^n}\sum_{q=1}^{m-1}\int_{T^*(\partial\Omega)\times\mathbf{R}}\mathrm{tr}\left\{L\left(\mathrm{grad}_\xi\Pi^q_+(\sigma)(\xi),\mathrm{grad}_\xi\Pi^{m-q}_-(\sigma)(\xi)\right)\right.$$
$$\left.-\Pi^q_+(\sigma)''(\xi)\Pi^{m-q}_-(\sigma)''(\xi)\cdot H\right\}\,d\xi\,dA.$$

Here $L(\operatorname{grad}_\xi \tau, \operatorname{grad}_\xi \tau)$ simply means $L_{ij}\tau^i\tau^j$ (summation convention). Also, H has the same meaning as in Theorem 1.1.

PROOF. By adding and subtracting

$$\operatorname{tr} k(x_1)\cdots k(x_{m-1})k(-x_1-\cdots-x_{m-1})\operatorname{vol}(\alpha\Omega)$$

to the integrand of (1.4) and using the fact that

$$\int\cdots\int k(x_1)\cdots k(x_{m-1})k(-x_1-\cdots-x_{m-1})\,dx_1\cdots dx_{m-1}$$
$$= k*\cdots *k(0) = \frac{1}{(2\pi)^n}\int \sigma^m(\xi)\,d\xi$$

(1.4) can be rewritten as

$$\operatorname{tr}(T_\alpha^m) = \left(\frac{\alpha}{2\pi}\right)^n \operatorname{vol}(\Omega)\int \operatorname{tr}\sigma^m(\xi)\,d\xi$$
$$(3.2)\qquad -\int\cdots\int \operatorname{tr} k(x_1)\cdots k(x_{m-1})k(-x_1-\cdots-x_{m-1})$$
$$\times\{\operatorname{vol}(\alpha\Omega) - \operatorname{vol}(\alpha\Omega\cap\alpha\Omega + x_1\cap\cdots$$
$$\cap\alpha\Omega + x_1 + \cdots + x_{m-1})\}\,dx_1\cdots dx_{m-1}.$$

By Lemma 2.5

$$\alpha^{-n+2}\Big|(\operatorname{vol}(\alpha\Omega) - \operatorname{vol}(\alpha\Omega\cap\alpha\Omega + x_1\cap\cdots\cap\alpha\Omega + x_1 + \cdots + x_{m-1}))$$
$$-\alpha^{n-1}\int_{\partial\Omega}\max(0, x_1\cdot n_x,\ldots,(x_1+\cdots+x_{m-1})\cdot n_x)\,dA\Big|$$
$$\leqslant C\max_i\left(|x_1+\cdots+x_1|^2\right) \leqslant C(m-1)^2\sum_{i=1}^{m-1}|x_i|^2.$$

Moreover, by Lemma 3.2,
(3.3)

$$C(m-1)^2\int\cdots\int\left|k(x_1)\cdots k(x_{m-1})k(-x_1-\cdots-x_{m-1})\sum_{i=1}^{m-1}|x_i|^2\right|dx_1\cdots dx_{m-1}$$
$$\leqslant C(m-1)^4\int|x|^2|k(x)|^2\,dx\|k\|_1^{m-2}.$$

Thus by the dominated convergence theorem

$$\lim_{\alpha\to\infty}\alpha^{-n+2}\int\cdots\int k(x_1)\cdots k(x_{m-1})k(-x_1-\cdots-x_{m-1})$$
$$\times\Big\{(\operatorname{vol}(\alpha\Omega) - \operatorname{vol}(\alpha\Omega\cap\alpha\Omega + x_1\cap\cdots$$
$$\cap\alpha\Omega + x_1 + \cdots + x_{m-1}))$$
$$-\alpha^{n-1}\int_{\partial\Omega}\max(0, x_1\cdot n_x,\ldots,(x_1+\cdots+x_{m-1})\cdot n_x)\,dA\Big\}\,dx_1\cdots dx_{m-1}$$

can be found by passing the limit under the integral. The result, by Corollary 2.4, is

$$\frac{1}{2}\int\cdots\int \operatorname{tr} k(x_1)\cdots k(x_{m-1})k(-x_1-\cdots-x_{m-1})$$
$$\times\sum_{q=1}^{m-1}\int_{\tilde S_q} L(x_{1_t}+\cdots+x_{q_t}, x_{1_t}+\cdots+x_{q_t})$$
$$-\big((x_1+\cdots+x_q)\cdot n_x\big)^2 H\,dA\,dx_1\cdots dx_{m-1},$$

where $\tilde S_q = \{x\in\partial\Omega\colon (x_1+\cdots+x_q)\cdot n_x > \max_{i\neq q}(0,(x_1+\cdots+x_i)\cdot n_x)\}$.

Thus the expression in brackets in (3.2) can be replaced by

$$\alpha^{n-1}\int_{\partial\Omega}\max(0, x_1\cdot n_x,\ldots,(x_1+\cdots+x_{m-1})\cdot n_x)\,dA$$
$$+\frac{\alpha^{n-2}}{2}\sum_{q=1}^{m-1}\int_{\tilde S_q} L(x_{1_t}+\cdots+x_{q_t}, x_{1_t}+\cdots+x_{q_t})$$
$$-\big((x_1+\cdots+x_q)\cdot n_x\big)^2 H\,dA$$

with error $o(\alpha^{n-2})$ and it follows that

$$a_1 = -\int\cdots\int \operatorname{tr} k(x_1)\cdots k(x_{m-1})k(-x_1-\cdots-x_{m-1}) \tag{3.4}$$
$$\times\int_{\partial\Omega}\max(0, x_1\cdot n_x,\ldots,(x_1+\cdots+x_{m-1})\cdot n_x)\,dA\,dx_1\cdots dx_{m-1},$$
$$a_2 = -\frac{1}{2}\int\cdots\int \operatorname{tr} k(x_1)\cdots k(x_{m-1})k(-x_1-\cdots-x_{m-1})$$
$$\times\sum_{q=1}^{m-1}\int_{\tilde S_q} L(x_{1_t}+\cdots+x_{q_t}, x_{1_1}+\cdots+x_{q_t})$$
$$-\big((x_1+\cdots+x_q)\cdot n_x\big)^2 H\,dA\,dx_1\cdots dx_{m-1}.$$

Next, $\tilde S_q = \{x\in\partial\Omega\colon$ for $s=1,\ldots,q$, $\Sigma_{r=s}^{q} x_r\cdot n_x > 0$ and for $t=1,\ldots,m-1-q$, $\Sigma_{r=q+1}^{q+t} x_r\cdot n_x < 0\}$. Change variables: For $s=1,\ldots,q$ let $z_s=\Sigma_{r=s}^{q} x_r$ and for $t=1,\ldots,m-1-q$ let

$$z_{q+t}=\sum_{r=q+1}^{q+t} x_r.$$

This gives

$$a_1 = -\sum_{q=1}^{m-1}\int_{\partial\Omega} dA\int\cdots\int z_1\cdot n_x \operatorname{tr} k(z_1-z_2)\cdots k(z_{q-1}-z_q)$$
$$\times k(z_q)k(z_{q+1})k(z_{q+2}-z_{q+1})\cdots k(z_{m-1}-z_{m-2})k(-z_1-z_{m-1})$$
$$\times\chi_+(z_1\cdot n_x)\cdots\chi_+(z_q\cdot n_x)\chi_-(z_{q+1}\cdot n_x)\cdots\chi_-(z_{m-1}\cdot n_x)dz_1\cdots dz_{m-1},$$

where

$$\chi_+(z\cdot n_x)=\begin{cases}1, & z\cdot n_x\geqslant 0,\\ 0, & z\cdot n_x<0,\end{cases}$$

and $\chi_-(z\cdot n_x)$ is defined in the obvious way. For a_2 the same expression is obtained except that $z_1\cdot n_x$ is replaced by $\frac{1}{2}\{L_{ij}z^i_{1_t}z^j_{1_t}-H(z_1\cdot n_x)^2\}$, where the L_{ij} are the coefficients of the second fundamental form L and $z^i_{1_t}$ is the ith component of the projection of z_1 onto $T_x\partial\Omega$. The interchange in the order of integration in the a_2 term is justified by Fubini's theorem since the application of the dominated convergence theorem implied the integrand of the expression for a_2 in (3.4) is L_1. In the case of a_1, the justification follows easily from Lemma 3.2.

Define

$$k_\pm(z)=k(z)\chi_\pm(z\cdot n_x). \tag{3.5}$$

Note that if $\sigma_\pm$ is defined by (3.1) then

$$k_\pm(z)=\frac{1}{(2\pi)^n}\int e^{i\xi\cdot z}\sigma_\pm(\xi)\,d\xi. \tag{3.6}$$

Next, define inductively

$$\begin{aligned} &K^1_+(z)=k_+(z), && K^1_-(z)=k_-(z),\\ &K^n_+(z)=\left(k*K^{n-1}_+\right)_+(z), && K^n_-(z)=\left(K^{n-1}_-*k\right)_-(z).\end{aligned} \tag{3.7}$$

Then

$$\begin{aligned} a_1&=-\sum_{q=1}^{m=1}\int_{\partial\Omega}\int z\cdot n_x\operatorname{tr}K^q_+(z)K^{m-q}_-(-z)\,dz\,dA,\\ a_2&=-\frac{1}{2}\sum_{q=1}^{m=1}\int_{\partial\Omega}\int\left\{L_{ij}z^i_tz^j_t-H(z\cdot n_x)^2\right\}\operatorname{tr}K^q_+(z)K^{m-q}_-(-z)\,dz\,dA.\end{aligned} \tag{3.8}$$

Now $z^i_tK^q_+(z)$ and $z^j_tK^{m-q}_-(-z)$ are in $L_2(\mathbf{R}^n)$ by Lemma 3.1. Moreover, using (3.6)

$$\left(z^j_tK^q_+(z)\right)^\wedge=i\frac{\partial}{\partial\xi^j}(K^q_+)^\wedge=i\frac{\partial}{\partial\xi^j}\Pi^q_+(\sigma).$$

Similar formulas hold for the $z\cdot n_x$ terms. Altogether,

$$a_1=-i\sum_{q=1}^{m-1}\int_{\partial\Omega}\int\operatorname{tr}\left(\Pi^q_+(\sigma)^n\right)^\vee(z)\left(\Pi^{m-q}_-(\sigma)\right)^\vee(-z)\,dz\,dA,$$

$$\begin{aligned} a_2=-\frac{1}{2}\sum_{q=1}^{m-1}\int_{\partial\Omega}\int\operatorname{tr}L_{ij}&\left(\Pi^q_+(\sigma)^i\right)^\vee(z)\left(\Pi^{m-q}_-(\sigma)^j\right)^\vee(-z)\\ &-\left(\Pi^q_+(\sigma)^n\right)^\vee(z)\left(\Pi^{m-q}_-(\sigma)^n\right)^\vee(-z)\cdot H\,dz\,dA.\end{aligned}$$

The result is now immediate from the general formula

$$\int k_1(z)k_2(-z)\,dz=\frac{1}{(2\pi)^n}\int\hat{k}_1(\xi)\hat{k}_2(\xi)\,d\xi.$$

PROOF OF LEMMA 3.1. The proof of (i) is given, the proof of (ii) being essentially the same.

$$\int |x|\left(|k|^{(m-1)}(x)\right)^2 dx$$
$$= \int\left\{\int \cdots \int |x - x_1 + x_1 - x_2 + x_2 - \cdots - x_{m-2}|^{1/2}\right.$$
$$\left. |k(x-x_1)|\,|k(x-x_2)| \cdots |k(x_{m-2})| dx_1 \cdots dx_{m-2}\right\}^2 dx$$
$$\leqslant \int\left\{\int \cdots \int \left(|x-x_1|^{1/2} + |x_1 - x_2|^{1/2} + \cdots + |x_{m-2}|^{1/2}\right)\right.$$
$$\left. |k(x-x_1)|\,|k(x_1-x_2)| \cdots |k(x_{m-2})| dx_1 \cdots dx_{m-2}\right\}^2 dx.$$

The terms in the integrand are all scalar valued and so commute. Thus the expression in brackets is the sum of $m-1$ terms, each of which is the convolution of $|x|^{1/2}|k(x)|$ and $m-2$ copies $|k(x)|$. That is, the above expression equals

$$\int\left\{(m-1)\int \cdots \int |x-x_1|^{1/2}|k(x-x_1)|\,|k(x_1-x_2)| \cdots \right.$$
$$\left. |k(x_{m-2})| dx_1 \cdots dx_{m-2}\right\}^2 dx$$
$$= (m-1)^2 \int \cdots \int\left\{\int |x-x_1|^{1/2}|k(x-x_1)|\,|x-z_1|^{1/2}|k(x-z_1)| dx\right\}$$
$$\times |k(x_1-x_2)| \cdots |k(x_{m-2})|\,|k(z_1-z_2)|$$
$$\cdots |k(z_{m-2})| dx_1 \cdots dx_{m-2}\, dz_1 \cdots dz_{m-2}.$$

Since

$$\int |x-x_1|^{1/2}|k(x-x_1)|\,|x-z_1|^{1/2}|k(x-z_1)| dx$$
$$\leqslant \left\{\int |x-x_1|\,|k(x-x_1)|^2 dx\right\}^{1/2}\left\{\int |x-z_1|\,|k(x-z_1)|^2 dx\right\}^{1/2}$$

the last expression is less than or equal to

$$(m-1)^2 \int |x|\,|k(x)|^2 dx \left\{\int |k(x)| dx\right\}^{2(m-2)}.$$

4. Proof of the main result. Throughout this section k will be scalar valued. The following version of the identity of Spitzer will be used. Using the notation $k^{(n)} = k * \cdots * k$, where, as before, $*$ denotes convolution on $\mathbf{R}^n$, define

$$E(k) - 1 = \sum_{n=1}^{\infty} \frac{k^{(n)}}{n!}$$

and

$$-L(1-f)=\sum_{n=1}^{\infty}\frac{k^{(n)}}{n}\quad \text{for } \|k\|_1<1.$$

Then with $K_+^n(z)$ defined as in (3.7), the identity is

$$\sum_{n=1}^{\infty}K_+^n(z)=E(-[L(1-k)]_+)-1,$$

the plus on the right side being taken in the sense of (3.5).

LEMMA 4.1. *If* $|\lambda|>\|k\|_1$ *and* L *is defined as above then* $L(1-k/\lambda)\in L_1(\mathbf{R}^n)\cap L_2(\mathbf{R}^n)$.

PROOF. If $\|\ \|$ denotes either the L_1 or L_2 norm then

$$\left\|\sum_{n=1}^{\infty}\frac{k^{(n)}}{n|\lambda|^n}\right\|\leqslant\sum_{n=1}^{\infty}\left\|\frac{k^{(n)}}{n|\lambda|^n}\right\|\leqslant\sum_{n=1}^{\infty}\frac{1}{n|\lambda|^n}\|k\|_1^{n-1}\|k\|$$
$$\leqslant\frac{1}{|\lambda|}\|k\|\sum_{n=1}^{\infty}\frac{\|k\|_1^{n-1}}{|\lambda|^{n-1}}.$$

LEMMA 4.2. (i) $\|\sigma^\beta\|_2\leqslant\|zk(z)\|_2$ *for any* $\beta=1,\dots,n$.
(ii) $\|(\log(1-\sigma/\lambda))^\beta\|_2\leqslant(\inf|\lambda-\sigma|)^{-1}\|\sigma^\beta\|_2$.

PROOF. (i) For instance, $\|\sigma^i\|_2=\|z_i^ik(z)\|_2\leqslant\|zk(z)\|_2$.
(ii) $(\log(1-\sigma/\lambda))^\beta=-\sigma^\beta/(\lambda-\sigma)$.

Since spec radius $(T_\alpha)\leqslant\|T_\alpha\|\leqslant\|k\|_1$, F is analytic on $\operatorname{spec}(T_\alpha)$ for all α. Let $\gamma(t)\subset\mathbf{C}$ be a circle of radius greater than $\|k\|_1$ centered at the origin and contained in the domain of analyticity of F. Then

$$(\lambda-T_\alpha)^{-1}=\sum_{m=0}^{\infty}\frac{T_\alpha^m}{\lambda^{m+1}}.$$

Hence

$$\operatorname{tr}F(T_\alpha)=\operatorname{tr}\frac{1}{2\pi i}\int_\gamma F(\lambda)(\lambda-T_\alpha)^{-1}\,d\lambda \tag{4.1}$$
$$=\operatorname{tr}\frac{1}{2\pi i}\int_\gamma F(\lambda)\left\{\frac{1}{\lambda}+\frac{T_\alpha}{\lambda^2}+\sum_{m=2}^{\infty}\frac{T_\alpha^m}{\lambda^{m+1}}\right\}d\lambda.$$

Since $F(0)=0$, the first integral is zero. Also,

$$\operatorname{tr}\frac{1}{2\pi i}\int_\gamma F(\lambda)\frac{T_\alpha}{\lambda^2}\,d\lambda=\left(\frac{\alpha}{2\pi}\right)^n\operatorname{vol}(\Omega)\frac{1}{2\pi i}\int_\gamma\int F(\lambda)\frac{\sigma(\xi)}{\lambda^2}\,d\xi\,d\lambda \tag{4.2}$$

since

$$\operatorname{tr}(T_\alpha) = \left(\frac{\alpha}{2\pi}\right)^n \operatorname{vol}(\Omega) \int \sigma(\xi)\, d\xi.$$

Consider now the sum in (4.1). From §3

$$\lim_{\alpha\to\infty} \alpha^{-n+2}\Big\{\operatorname{tr}(T_\alpha^m) - \left(\frac{\alpha}{2\pi}\right)^n \operatorname{vol}(\Omega)\int \sigma^m(\xi)\, d\xi + \alpha^{n-1}\sum_{q=1}^{m-1}\int\int z\cdot n_x K_+^q(z) K_-^{m-q}(-z)\, dz\, dA\Big\}$$
$$= -\frac{1}{2}\sum_{q=1}^{m-1}\int\int\left\{L_{ij}z_t^i z_t^j - (z\cdot n_x)^2 H\right\}K_+^q(z)K_-^{m-q}(-z)\, dz\, dA.$$

First using (3.2) and comparing the expression for a_1 in (3.4) and (3.8) and then using (3.3) gives, for all $\alpha \geqslant 1$,

$$\alpha^{-n+2}\left|\operatorname{tr}(T_\alpha^m) - \left(\frac{\alpha}{2\pi}\right)^n \operatorname{vol}(\Omega)\int\sigma^m(\xi)\, d\xi + \alpha^{n-1}\sum_{q=1}^{m-1}\int\int z\cdot n_x K_+^q(z)K_-^{m-q}(-z)\, dz\, dA\right|$$
$$= \alpha^{-n+2}\Big|\int\cdots\int k(x_1)\cdots k(x_{m-1})k(-x_1-\cdots-x_{m-1})$$
$$\times\Big\{-(\operatorname{vol}(\alpha\Omega) - \operatorname{vol}(\alpha\Omega\cap\alpha\Omega + x_1\cap\cdots\cap\alpha\Omega + x_1+\cdots+x_{m-1}))$$
$$+\alpha^{n-1}\int_{\partial\Omega}\max(0, x_1\cdot n_x,\ldots,(x_1+\cdots+x_{m-1})\cdot n_x)\, dA\Big\}\, dx_1\cdots dx_{m-1}\Big|$$
$$\leqslant C\int|x|^2|k(x)|^2\, dx(m-1)^4\|k\|_1^{m-2}.$$

Since

$$\int_\gamma |F(\lambda)|\sum_{m=2}^{\infty}\frac{1}{|\lambda|^{m+1}}(m-1)^4\|k\|_1^{m-2}\, d\lambda < \infty$$

if $\|k\|_1/|\lambda| < 1$ it follows from the above limit and the dominated convergence theorem that

(4.3)
$$\lim_{\alpha\to\infty}\alpha^{-n+2}\Big[\frac{1}{2\pi i}\int_\gamma F(\lambda)\sum_{m=2}^{\infty}\frac{1}{\lambda^{m+1}}\Big\{\operatorname{tr} T_\alpha^m - \left(\frac{\alpha}{2\pi}\right)^n\operatorname{vol}(\Omega)\int\sigma^m(\xi)\, d\xi + \alpha^{n-1}\sum_{q=1}^{m-1}\int\int z\cdot n_x K_+^q(z)K_-^{m-q}(-z)\, dz\, dA\Big\}\, d\lambda\Big]$$
$$= -\frac{1}{2}\frac{1}{2\pi i}\int_\gamma F(\lambda)\sum_{m=2}^{\infty}\frac{1}{\lambda^{m+1}}\int\int\left\{L_{ij}z_t^i z_t^j - (z\cdot n_x)^2 H\right\}\times\sum_{q=1}^{m-1}K_+^q(z)K_-^{m-q}(-z)\, dz\, dA\, d\lambda.$$

From (4.1), (4.2) and (4.3)

$$\operatorname{tr} F(T_\alpha) = \left(\frac{\alpha}{2\pi}\right)^n \operatorname{vol}(\Omega)\frac{1}{2\pi i}\int\int F(\lambda)\sum_{m=1}^{\infty}\frac{\sigma^m(\xi)}{\lambda^{m+1}}\,d\xi\,d\lambda$$

$$-\alpha^{n-1}\frac{1}{2\pi i}\int\int\int F(\lambda) z\cdot n_x \sum_{m=2}^{\infty}\frac{1}{\lambda^{m+1}}\sum_{q=1}^{m-1}K_+^q(z) \times K_-^{m-q}(-z)\,dz\,dA\,d\lambda$$

$$(4.4)\qquad -\frac{\alpha^{n-2}}{2}\frac{1}{2\pi i}\int\int\int F(\lambda)\left\{L_{ij}z_t^i z_t^j - (z\cdot n_x)^2 H\right\} \times \sum_{m=2}^{\infty}\frac{1}{\lambda^{m+1}}\sum_{q=1}^{m-1}K_+^q(z)K_-^{m-q}(-z)\,dz\,dA\,d\lambda$$

$$+o(\alpha^{n-2}),\qquad \alpha\to\infty,$$

where Lemma 3.2 is used to justify interchanging the infinite sums and integrals.

The first term on the right side is

$$(4.5)\qquad \left(\frac{\alpha}{2\pi}\right)^n \operatorname{vol}(\Omega)\int F(\sigma(\xi))\,d\xi.$$

The change in the order of integration is justified as follows:

$$\sum_{m=1}^{\infty}\frac{\sigma^m}{\lambda^{m+1}} = \frac{1}{\lambda}\sum_{m=0}^{\infty}\frac{\sigma^m}{\lambda^m} - \frac{1}{\lambda} = \frac{1}{\lambda}\left(1-\frac{\sigma}{\lambda}\right)^{-1} - \frac{1}{\lambda} = \frac{\sigma}{\lambda(\lambda-\sigma)}$$

and the L_1 norm of this is bounded by

$$|\lambda|^{-1}(\inf|\lambda-\sigma|)^{-1}\|\sigma\|_1.$$

To handle the other terms note that

$$\sum_{m=2}^{\infty}\frac{1}{\lambda^{m+1}}\sum_{q=1}^{m-1}K_+^q(z)K_-^{m-q}(-z)$$

$$(4.6)\qquad = \frac{1}{\lambda}\sum_{r=1}^{\infty}\frac{K_+^r(z)}{\lambda^r}\sum_{s=1}^{\infty}\frac{K_-^s(-z)}{\lambda^s}$$

$$= \frac{1}{\lambda}\left(E\left(-\left[L\left(1-\frac{k}{\lambda}\right)\right]_+\right)(z) - 1\right)\left(E\left(-\left[L\left(1-\frac{k}{\lambda}\right)\right]_-\right)(-z) - 1\right)$$

by Spitzer's identity. By Lemma 4.1 the expression $(-L(1-k/\lambda))^\wedge_\pm$ makes sense and

$$(4.7)\qquad \left(E(-[L(1-k/\lambda)]_\pm) - 1\right)^\wedge = \exp(-\log(1-\hat{k}/\lambda))_\pm - 1 = \tau^\pm - 1,$$

where the notation $\tau^\pm$ has been introduced for $\exp(\log(1-\sigma/\lambda)^{-1})_\pm$.

Using (4.4), (4.6), and (4.7) and then integrating by parts twice with respect to η

yields

$$a_1 = -\frac{1}{2\pi i}\int\int\int \frac{F(\lambda)}{\lambda} z \cdot n_x(\tau^+ - 1)^\vee(z)(\tau^- - 1)^\vee(-z)\,dz\,dA\,d\lambda$$

$$= -\frac{1}{2\pi i}\frac{1}{(2\pi)^n}\int\int\int \frac{F(\lambda)}{\lambda} i(\tau^+)^n(\xi)(\tau^- - 1)(\xi)\,d\xi\,dA\,d\lambda$$

$$= -\frac{1}{2\pi i}\frac{1}{(2\pi)^n}\int\int\int \frac{F(\lambda)}{\lambda} i\tau^+(\xi)(\tau^-)^n(\xi)\,d\xi\,dA\,d\lambda \tag{4.8}$$

$$= -\frac{1}{2\pi i}\frac{1}{(2\pi)^n}\int\int\int \frac{F(\lambda)}{\lambda} i(\tau^+)^n(\xi)\tau^-(\xi)\,d\xi\,dA\,d\lambda. \tag{4.9}$$

For any $\beta = 1,\dots,n$

$$(\tau^\pm)^\beta = \tau^\pm\left(\log(1-\sigma/\lambda)^{-1}\right)^\beta_\pm. \tag{4.10}$$

By Lemma 4.2 the expression $(-\log(1-\sigma/\lambda))^\beta_\pm$ makes sense and one has almost everywhere (using the definition of $\tilde{\ }$)

$$\left(\log(1-\sigma/\lambda)^{-1}\right)^\beta_\pm = \left(\sigma^\beta/(\lambda-\sigma)\right)_\pm. \tag{4.11}$$

Also,

$$\tau^+\tau^- = \lambda(\lambda-\sigma)^{-1}. \tag{4.12}$$

Thus, (4.8) can be rewritten as

$$a_1 = \frac{1}{(2\pi)^n}\frac{1}{2\pi i}\int\int\int F(\lambda)i(\lambda-\sigma)^{-1}\left(\frac{\sigma^n}{\lambda-\sigma}\right)_-\,d\xi\,dA\,d\lambda$$

and (4.9) can be written as

$$a_1 = -\frac{1}{(2\pi)^n}\frac{1}{2\pi i}\int\int\int F(\lambda)i(\lambda-\sigma)^{-1}\left(\frac{\sigma^n}{\lambda-\sigma}\right)_+\,d\xi\,dA\,d\lambda.$$

Since in general $\sigma_- - \sigma_+ = -i\tilde{\sigma}$ these two expressions can be added to give

$$a_1 = \frac{1}{2}\frac{1}{(2\pi)^n}\frac{1}{2\pi i}\int\int\int F(\lambda)(\lambda-\sigma)^{-1}\left(\frac{\sigma^n}{\lambda-\sigma}\right)^{\sim}d\xi\,dA\,d\lambda \tag{4.13}$$

$$= \frac{1}{2}\frac{1}{(2\pi)^n}\frac{1}{2\pi i}\int\int\int\frac{1}{\pi}\int F(\lambda)(\lambda-\sigma(\eta))^{-1}\frac{\sigma^n(\zeta)}{\lambda-\sigma(\zeta)}\frac{d\zeta}{\zeta-\eta}d\xi\,dA\,d\lambda$$

$$= \frac{1}{4\pi}\frac{1}{(2\pi)^n}\int\int\int\frac{F(\sigma(\eta))-F(\sigma(\zeta))}{\sigma(\eta)-\sigma(\zeta)}\frac{\sigma^n(\zeta)-\sigma^n(\eta)}{\zeta-\eta}d\zeta\,d\xi\,dA.$$

The interchange in the order of integration is justified as follows.

$$\int\left(\frac{\sigma^n}{\lambda-\sigma}\right)^{\sim}d\xi = \int\left(-\log\left(1-\frac{\sigma}{\lambda}\right)\right)^{n\sim}d\xi = \int\left(-\log\left(1-\frac{\sigma}{\lambda}\right)\right)^{\sim n}d\xi = 0.$$

Thus

$$\int(\lambda-\sigma)^{-1}\left(\frac{\sigma^n}{\lambda-\sigma}\right)^{\sim}d\xi=\int\left\{(\lambda-\sigma)^{-1}-\frac{1}{\lambda}\right\}\left(\frac{\sigma^n}{\lambda-\sigma}\right)^{\sim}d\xi \tag{4.14}$$

$$=\int\frac{1}{\lambda}\frac{\sigma}{\lambda-\sigma}\left(\frac{\sigma^n}{\lambda-\sigma}\right)^{\sim}d\xi$$

and using Lemma 4.2 and the fact that the Hilbert transform is norm preserving

$$\int\int\int\left|\frac{F(\lambda)}{\lambda}\frac{\sigma}{\lambda-\sigma}\left(\frac{\sigma^n}{\lambda-\sigma}\right)^{\sim}\right|d\xi\,dA\,d\lambda$$

$$\leqslant\int_\gamma\int_{\partial\Omega}\frac{|F(\lambda)|}{|\lambda|}\left\|\frac{\sigma}{\lambda-\sigma}\right\|_2\left\|\frac{\sigma^n}{\lambda-\sigma}\right\|_2 dA\,d\lambda$$

$$\leqslant|\lambda|^{-1}(\inf|\lambda-\sigma|)^{-2}\|\sigma\|_2\,\|zk(z)\|_2\operatorname{vol}(\partial\Omega)\int_\gamma|F(\lambda)|d\lambda$$

$$<\infty.$$

Thus $d\lambda$ may be interchanged with $d\xi\,dA$. To interchange $d\lambda$ with $d\zeta$ note that for fixed η the change of variable $\bar\zeta=\zeta-\eta$ gives

$$\lim_{\varepsilon\to0}\int_{|\zeta-\eta|>\varepsilon}\frac{\sigma^n(\zeta)}{\lambda-\sigma(\zeta)}\frac{1}{\zeta-\eta}d\zeta=\lim_{\varepsilon\to0}\int_{|\zeta|>\varepsilon}\frac{\sigma^n(\zeta+\eta)}{\lambda-\sigma(\zeta+\eta)}\frac{1}{\zeta}d\zeta$$

$$=\lim_{\varepsilon\to0}\int_\varepsilon^\infty\left\{\frac{\sigma^n(\eta+\zeta)}{\lambda-\sigma(\eta+\zeta)}-\frac{\sigma^n(\eta-\zeta)}{\lambda-\sigma(\eta-\zeta)}\right\}\frac{1}{\zeta}d\zeta.$$

The integrand in the last expression is continuous for $\zeta\neq0$ and

$$\lim_{\zeta\to0^+}\frac{1}{\zeta}\left\{\frac{\sigma^n(\eta+\zeta)}{\lambda-\sigma(\eta+\zeta)}-\frac{\sigma^n(\eta-\zeta)}{\lambda-\sigma(\eta-\zeta)}\right\}=0$$

since σ is twice differentiable. Hence the integration with respect to ζ is actually proper at $\zeta=0$ and the integrand is L_1 if the domain of integration is restricted to $0\leqslant\zeta\leqslant1$. On the other hand, for $1<\zeta<\infty$

$$\int_\gamma\int_1^\infty\left|F(\lambda)(\lambda-\sigma(\eta))^{-1}\frac{\sigma^n(\eta+\zeta)}{\lambda-\sigma(\eta+\zeta)}\frac{1}{\zeta}\right|d\zeta\,d\lambda$$

$$\leqslant\|\sigma^n\|_2\left\|\frac{1}{\zeta}\right\|_2(\inf|\lambda-\sigma|)^{-2}\int_\gamma|F(\lambda)|d\lambda<\infty.$$

Now for a_2. From (4.4), (4.6) and (4.7)

$$a_2=-\frac{1}{2}\frac{1}{2\pi i}\int\int\int\frac{F(\lambda)}{\lambda}\left[L_{ij}z_t^iz_t^j-(z\cdot n_x)^2H\right]$$

$$\times(\tau^+-1)^\vee(z)(\tau^--1)^\vee(-z)\,dz\,dA\,d\lambda$$

$$=-\frac{1}{2}\frac{1}{(2\pi)^n}\frac{1}{2\pi i}\int\int\int\frac{F(\lambda)}{\lambda}\{L_{ij}(\tau^+)^i(\xi)(\tau^-)^j(\xi)$$

$$-H(\tau^+)^n(\xi)(\tau^-)^n(\xi)\}\,d\xi\,dA\,d\lambda.$$

From (4.10), (4.11) and (4.12) this equals

$$a_2 = -\frac{1}{2}\frac{1}{(2\pi)^n}\frac{1}{2\pi i}\int\int\int\frac{F(\lambda)}{\lambda-\sigma(\xi)}\left\{L_{ij}\left(\frac{\sigma^i}{\lambda-\sigma}\right)_+(\xi)\left(\frac{\sigma^j}{\lambda-\sigma}\right)_-(\xi)\right.$$
$$\left.-H\left(\frac{\sigma^n}{\lambda-\sigma}\right)_+(\xi)\left(\frac{\sigma^n}{\lambda-\sigma}\right)_-(\xi)\right\}\,d\xi\,dA\,d\lambda.$$

Writing out the Hilbert transforms explicitly and then performing a straightforward residue calculation yields

$$(4.15)\quad a_2 = -\frac{1}{16}\frac{1}{(2\pi)^n}\int\int F''(\sigma(\xi))\left\{L_{ij}\sigma^i(\xi)\sigma^j(\xi) - H(\sigma^n(\xi))^2\right\}\,d\xi\,dA$$
$$-\frac{1}{8\pi^2}\frac{1}{(2\pi)^n}\int\int\int\int\left\{\frac{F(\sigma(\eta))}{(\sigma(\eta)-\sigma(\zeta_1))(\sigma(\eta)-(\zeta_2))}\right.$$
$$+\frac{F(\sigma(\zeta_1))}{(\sigma(\zeta_1)-\sigma(\eta))(\sigma(\zeta_1)-\sigma(\zeta_2))}$$
$$\left.+\frac{F(\sigma(\zeta_2))}{(\sigma(\zeta_2)-\sigma(\eta))(\sigma(\zeta_2)-\sigma(\zeta_1))}\right\}$$
$$\times\left(L_{ij}\sigma^i(\zeta_1)\sigma^j(\zeta_2) - H\sigma^n(\zeta_1)\sigma^n(\zeta_2)\right)\frac{d\zeta_1}{\zeta_1-\eta}\frac{d\zeta_2}{\zeta_2-\eta}\,d\xi\,dA.$$

It will be shown that the first term in the above expression is zero. The theorem then follows from (4.4), (4.5), (4.14) and (4.15). It is enough to show the term is zero in the case F is a power, $F(\lambda) = \lambda^m$, since in the general case F'' is analytic on the range of σ. For $F(\lambda) = \lambda^m$ the expression in question is

$$-\frac{1}{16}\frac{1}{(2\pi)^n}\int_{\partial\Omega}\int m(m-1)(\sigma(\xi))^{m-2}\left\{L_{ij}\sigma^i\sigma^j - H(\sigma^n)^2\right\}\,d\xi\,dA$$
$$= -\frac{1}{16}\frac{1}{(2\pi)^n}\int_{\partial\Omega}\int m\left\{L_{ij}(\sigma^{m-1})^i\sigma^j - H(\sigma^{m-1})^n\sigma^n\right\}\,d\xi\,dA$$
$$= -\frac{m}{16}\int_{\partial\Omega}\int\left\{L_{ij}z_t^iz_t^j - (z\cdot n_x)^2H\right\}k^{(m-1)}(z)k(-z)\,dz\,dA.$$

It is easy to justify interchanging the order of integration and the result then follows from an identity discovered by Widom and J. Dadok:

$$(4.16)\qquad \int_{\partial\Omega} L(z_t, z_t) - (z\cdot n_x)^2H\,dA = 0.$$

The following proof is the result of communications with R. Osserman, L. Simon, and B. Lawson. It is known that if X is a vector field along M, a closed smooth hypersurface in $\mathbf{R}^n$, then

$$(4.17)\qquad \int_M\sum_{i=1}^{n-1}\langle\nabla_{\tau_i}X, \tau_i\rangle\,dA(x) = \int_M\langle X, n_x\rangle H\,dA(x),$$

where the τ_i are an orthonormal basis for the tangent space to M at x, ∇_v denotes covariant differentiation in the direction of v, and the brackets denote the ordinary $\mathbf{R}^n$ inner product. To prove this note

$$\sum_i \langle \nabla_{\tau_i} X, \tau_i \rangle = \sum_i \langle \nabla_{\tau_i} \langle X, n_x \rangle n_x, \tau_i \rangle + \sum_i \langle \nabla_{\tau_i} X_t, \tau_i \rangle.$$

The integral over M of the second term is zero by the divergence theorem (see [**4**, pp. 188 and 193]) and the first term equals

$$\langle X, n_x \rangle \sum_i \langle \nabla_{\tau_i} n_x, \tau_i \rangle = \langle X, n_x \rangle H.$$

To prove (4.16) take $X = \langle z, n_x \rangle z$ in (4.17). The right side immediately gives (with $M = \partial\Omega$)

$$\int_{\partial\Omega} \langle z, n_x \rangle^2 H \, dA.$$

The left side gives

$$\int_{\partial\Omega} \sum_i \langle z, \nabla_{\tau_i} n_x \rangle \langle z, \tau_i \rangle \, dA.$$

Since $\nabla_{\tau_i} n_x = W_{n_x}(\tau_i)$, where W_{n_x} is the Weingarten map and the latter is selfadjoint, the integrand of this expression equals

$$\sum_i \langle W_{n_x}(z_t), \tau_i \rangle \langle z, \tau_i \rangle = \langle W_{n_x}(z_t), z_t \rangle = L(z_t, z_t).$$

5. The logarithm. Let $F(\lambda) = \log(1 + \lambda)$ and let γ be the circle of radius r about the origin with $\|k\|_1 < r < 1$. The following method of deriving the expression for a_1 in (1.2) from the expression in the main result was shown to the author by Widom. Justification of the various steps is accomplished as in §4.

From (4.14) and the first line of (4.13)

$$\begin{aligned} a_1 &= \frac{1}{2} \frac{1}{(2\pi)^n} \frac{1}{2\pi i} \int_\gamma \int_{\partial\Omega} \int \log(1 + \lambda) \frac{\sigma}{\lambda(\lambda - \sigma)} \left(\frac{\sigma^n}{\lambda - \sigma} \right)^{\sim} d\xi \, dA \, d\lambda \\ &= \frac{1}{2} \frac{1}{2\pi i} \int_\gamma \int_{\partial\Omega} \int \log(1 + \lambda) |z \cdot n_x| s'_\lambda(z) s_\lambda(-z) \, dz \, dA \, d\lambda, \end{aligned}$$

where $s_\lambda(z) = (\log(1 - \sigma/\lambda))^\vee(z)$ and the prime denotes differentiation with respect to λ. Since $|z \cdot n_x|$ is invariant under changing z to $-z$ it follows that

$$\begin{aligned} a_1 &= \frac{1}{4} \frac{1}{2\pi i} \int_\gamma \int_{\partial\Omega} \int \log(1 + \lambda) |z \cdot n_x| \frac{d}{d\lambda} (s_\lambda(z) s_\lambda(-z)) \, dz \, dA \, d\lambda \\ &= -\frac{1}{4} \frac{1}{2\pi i} \int_\gamma \int_{\partial\Omega} \int \frac{1}{\lambda + 1} |z \cdot n_x| s_\lambda(z) s_\lambda(-z) \, dz \, dA \, d\lambda. \end{aligned}$$

Let $\tilde{\gamma}$ be a circle centered at -1 with radius less than $1 - r$. Then

$$a_1 = \frac{1}{4} \frac{1}{2\pi i} \int_{\tilde{\gamma}} \int_{\partial\Omega} \int \frac{1}{\lambda + 1} |z \cdot n_x| s_\lambda(z) s_\lambda(-z) \, dz \, dA \, d\lambda$$

and performing the integration with respect to λ gives the result.

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